Mathematics Lessons from regular Floor Tilings

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Regular floor tiling patterns can be used to motivate a variety of mathematical discussions and investigations for students through a wide age range. In this paper, I discuss material I have used directly with children from 8 years of age up to 15 years. Some of the ideas could be adapted for use with younger or older students also. I consider it very desirable for the students to view the floor tilings in situ whenever possible. Some of the ideas discussed here have come from viewing the tilings from different angles, which is not possible working simply with photos or drawings. This will also help younger children to think about the differences angle and perspective make to how we look at a pattern. The mathematical areas covered include proportion and fractions; symmetry; number patterns; deriving algebraic formulae; programming.

Keywords: tiling, tessellation, algebraic pattern, symmetry, proportion, programming, Logo, geometry

1. Proportion, fractions and symmetry

Patterned floor tilings can be used to motivate work on proportion, fractions and symmetry, and will lead to rich mathematical discussions. The ideas in this section have been successfully used with several groups of children aged 8 to 11. Simplifying the pattern unit further would make these ideas accessible to children from about 6 or 7 upwards.

Figure 1. Floor tiling from the Lady Chapel, Ely Cathedral, UK

Ely Cathedral is a medieval building, and the Lady Chapel dates from the 14th century. The floor tiles are modern, but use designs found in the entrance to the West Door of the Cathedral which date from the 19th century (Figure 1).

Ideally the children should see floor tilings in situ, and be able to view them from a variety of angles, discussing the shapes and patterns that they see. If this is then followed up with photos when the class is back in school, interesting questions about orientation and perspective can be discussed. If a site visit is not possible, then it will be necessary to work just from photos. Making similar designs on a table top with card tiles will help children to see that the angle from which a pattern is viewed does not actually change the pattern, although it may change our perspective. They can also investigate whether rotating a pattern changes it in any fundamental way.

After an initial discussion, the pattern unit shown in Figure 2 was used for more detailed follow-up work, with a

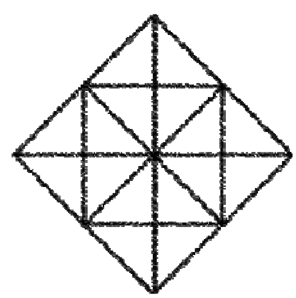

Figure 2. Pattern unit

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worksheet (which can be downloaded from http://motivate.maths.org/teachers/MathsArt/PatternsNumbersWorksheet.pdf) with six outlines of the unit on it.

The first activity for the children was to colour one outline so that it looked like a section of the real floor or photo. This is not a trivial activity for many children, requiring them to look carefully at what they see and to record it on a diagram, which is abstracted from reality, and which uses a half size triangular tile.

We then discussed what fraction and what proportion of the triangles were in dark and light colours. Once everyone was happy that half the triangular tiles were light coloured and half were dark coloured, and that in this pattern unit we need 8 out of 16 triangular tiles to be in each colour, the next activity was for them to colour in the remaining seven outlines so that in each case (a) exactly half the triangular tiles were light and half were dark, and (b) each colouring was different from all the others.

Once children had completed this task, we reviewed their designs. The first question we discussed was whether different children had produced designs the same as each other. This raised issues about whether the inverse colour scheme is the same or not and whether the same design rotated through 90 or 180 degrees is the same or not.

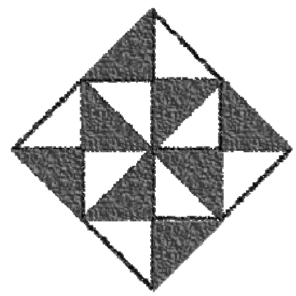

Figure 3. Tiling pattern with half the tiles coloured dark and half light

We then looked at the symmetry their coloured designs showed. How many lines of symmetry are possible? It appeared that you can have none, 1, 2 or 4, but not 3. Why is that? Could there be more than four lines of symmetry, and if not, why not? And what about a design like Figure 3? It has no lines of symmetry, but it appears to be symmetrical. How can we describe this?

These discussions focused on a pattern in which half the tiles are light and half dark. One group of children also discussed why the pattern was half light and half dark, considering aesthetic and symbolic issues in a church setting, and the use of 'light' and 'dark' as metaphors. This discussion arose from a chance remark by a teacher with this particular group, and was a particularly interesting development of the general theme.

The activities described above could be further extended to patterns in which some other proportion was coloured dark or light, such as a third, or a quarter or an eighth, or other fractions of the whole. Like the symmetry discussion, this raises questions about what fractions can be shown on a pattern like the one used, and why this is so.

2. Number patterns and sequences (pre-algebra)

The floor tiling in the Choir of Ely Cathedral (Figure 4) has been used with several groups of children in the 9-12 age range for work on number sequences. Viewing a pattern like this in situ means that questions can be asked about the edges of such a design – how did the tiler finish the pattern when they got to the edge of the floor? Could you do it differently?

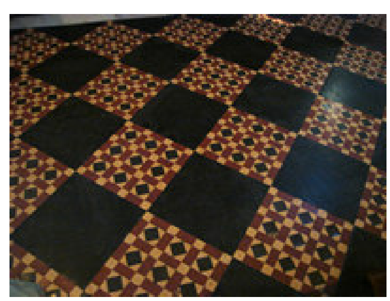

Figure 4. The Choir , Ely Cathedral, UK

None are visible on this photo, but there are also places where smaller units of this pattern can be found as a way of fitting it into the space available, and these provide a context for looking at number sequences. This can be motivated by asking children to imagine they are the tiler – how many of each type of tile do they need for a given space? If tiles have to be made to order, it is important to have enough, but too many would be wasteful.

Again the basic pattern can be abstracted for students to work on (Figure 5**Error! Reference source not found.**, and the worksheet can be downloaded from http://motivate.maths.org/teachers/MathsArt/PatternsNumbersWorksheet.pdf). For each diagram, students were first asked how many of the large black squares there would be – after ensuring they understood that the shape that looks like a diamond is in fact square. Then they worked on the numbers of small squares, rectangles and triangles (exemplars coloured in Figure 5).

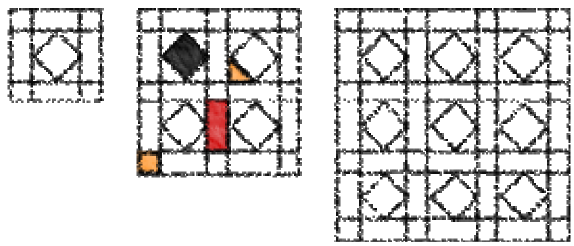

Figure 5. Tiling pattern, Choir, Ely Cathedral, UK

Students who do not yet have any knowledge of algebra can still be asked to think about how these numbers are related to the diagram number, what is special about the number sequence each shape gives, and how many of each shape the next diagram up might require. After the children, had worked out how many of each shape there were in the three patterns shown, I asked them to predict how many of each shape tile the fourth pattern might require. We then discussed their suggestions and their reasons for making them, then added a row and column to the third diagram, to check visually who was right.

Most had noticed that the large squares give a sequence of square numbers (1, 4, 9, …) and that the small squares also give a sequence of square numbers, but starting from 4 rather than 1. The sequence for the rectangles is harder, but on prompting they were able to tell me that the numbers form a sequence of multiples of 4 (4, 12, 24, …). This sequence is particularly interesting, and can be used to challenge the brightest students. What will the next value in the sequence be? (It is 40). How can we predict what the next multiple of 4 will be? (This sequence is 4 multiplied by the triangle numbers, ie. 1×4, 3×4, 6×4, 10×4, …). This raises the question as to where the four comes from and why we have the triangle numbers there.

Identifying the number sequence for the triangles (4, 16, 36, …) is also difficult working just from the numbers, but looking at the diagrams makes it obvious that there are four triangles for each large square, so we have a sequence which is four times the square numbers, ie. 1×4, 4×4, 9×4, ….

In all the groups who have done this particular activity, there were children able to explain the patterns they had observed verbally, often giving different perspectives from those I had noticed. Almost all children were able to give a reason for why we might expect to find square numbers and multiples of 4 in such a tiling pattern.

3. Number patterns and sequences (using algebra)

Any of these number sequences could also be used with older students who are learning algebra. Having found descriptions of the sequences, the question is then to write them as algebraic formulae, so that they can predict how many of each individual tile would be required for any size pattern. There are other floor patterns from the Ely Lady Chapel (Figure 6) which could be also be used in this way. This particular pattern is visually very simple, but like the more complicated patterns, it can be made mathematically very rich.

The lines drawn onto Figure 6 focus the attention on a particular sequence of stages of the pattern. The number of additional black (or grey) tiles at each stage gives a sequence of odd numbers (1, 3, 5, 7, 9, …), and the total number of black triangles at each stage is a square number (1, 4, 9, 16, 25, …).

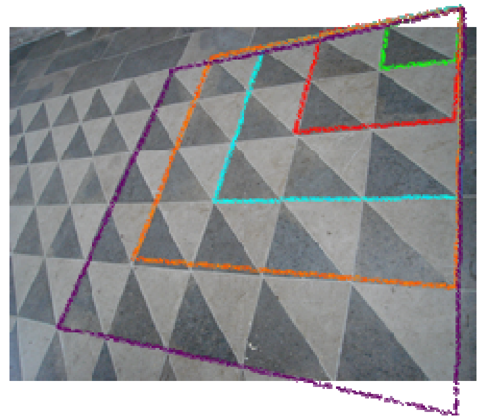

Figure 6. Floor tiling, Lady Chapel, Ely Cathedral, UK

Students could be asked to find other ways to demonstrate this result using small card tiles, and then to describe their results algebraically. They should also be challenged to explain the results. Why is it the next odd number which is added at each stage? Why is the total number of tiles always a square number? Are there other ways that these particular number sequences can be demonstrated in a tiling pattern? Having found formulae for either the black or the grey tiles, how do our formulae change if we consider the black and the grey tiles together?

Viewing this section of floor from a different perspective also allows the triangle numbers to be observed visually (Figure 7). Lines are again added to the photo to focus attention this time on a triangular pattern. The number of either black or grey tiles added at each stage is the next whole number (1, 2, 3, 4, …) – why is it different from what we observed with a square pattern? If we look at the total number of triangles of the given colour, then we have the sequence of triangle numbers (1, 3, 6, 10, 15, …). How does this relate to the square number pattern we observed earlier? Can we write algebraic formulae to describe these patterns? And again, if we consider the number of black and grey tiles together, how does that change the formulae?

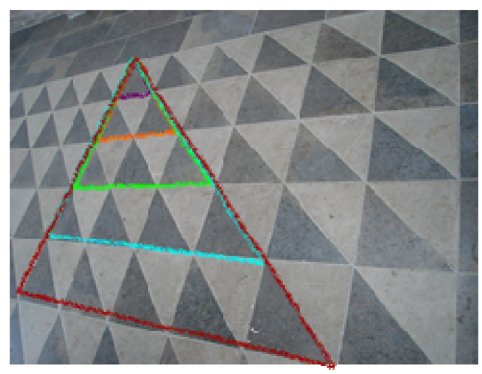

Figure 7. Floor tiling, Lady Chapel, Ely Cathedral, UK

4. Cosmati tilings

Westminster Cathedral in London, which is a twentieth century Roman Catholic cathedral, has floor tilings in the style of the Cosmati tiling designs (Figure 8) first developed in medieval Italy. The name Cosmati is that of the Roman family who first created inlaid ornamental mosaic designs, using marble from ancient Roman ruins, and arranging the fragments in geometric patterns. They rapidly developed a distinctive style. Similar medieval designs are found elsewhere in Europe, and in twentieth century work also. These designs can be used with students in the 12-15 age range for a number of investigative tasks.

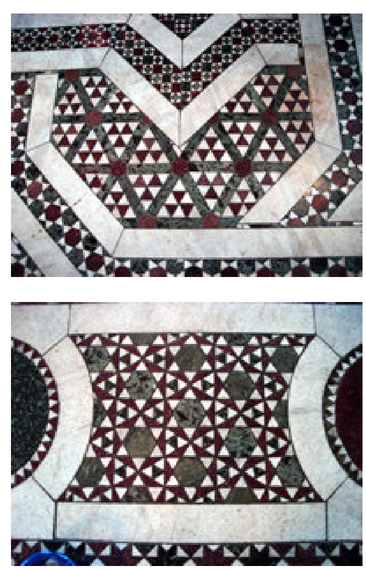

Figure 8. Cosmati tilings, Westminster Cathedral, London

One starting point is to give students a basic motif, such as those in Figure 9, which they can use to tessellate an area. Both motifs can be tessellated in either a linear or a radial direction, giving different designs, and with different challenges to consider about how the edge of a design will be defined.

The question they should then investigate is that of finding formulae for the numbers of each type of tile of which the motif is comprised for a given floor area. The difficulty of this activity will depend on the motif chosen and the shape of the final area, so the task can be made easier or harder, as required.

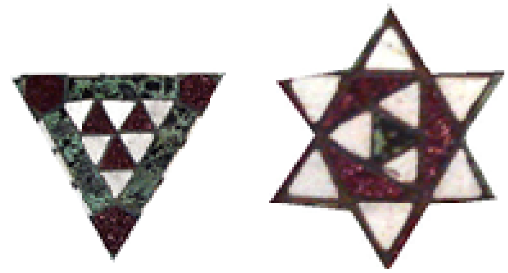

Figure 9. Motifs

5. Using software to create tilings: Thinking geometrically

Computer programs can also be used to explore tiling patterns with older students. The programming language, Logo, is ideal for this, and it is freely available from http://www.softronix.com/logo.html. There are versions for younger children which use 'turtles', the name given to the cursor (presumably because it looks a bit like a turtle!). All the programs associated with article are available at http://motivate.maths.org/teachers/teachers.php#topics and the Logo webpages include links to sites which introduce Logo and give more detail about the commands and putting them together to produce programs (known as Procedures).

Basic commands include drawing a line of a given length in a forward or backward direction, and turning through a specified angle. These can be built into regular polygons using the Repeat command. Creating a Procedure to draw a particular shape is a way of building up a library of programs which can then be used in other programs. Using a variable for a side length means that figures can then be drawn in any size. This also helps students develop their concept of an algebraic variable.

Suppose we want to create patterns based on the motifs in Figure 9. Both are composed of equilateral triangles, so a first step might well be to create a Procedure to draw an equilateral triangle (Figure 10). The left-hand version gives the basic program commands in a correct sequence (fd means 'forward', rt means 'turn right' through the specified angle, ht means 'hide turtle' or cursor, :*a* is a variable length, for which a suitable side length needs to be substituted when the program is executed, for instance, triangle 200 [no colon required] would give a triangle with side length 200 pixels). The right-hand version uses a Repeat command, and is thus more economical.

```
to triangle :a
rt 30
fd :a rt 120
fd :a rt 120
fd :a rt 120
lt 30
ht
end
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```
to triangle :a
rt 30
repeat 3[fd :a rt 120]
lt 30
ht
end
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Figure 10. Logo programs to create an equilateral triangle

This is a very simple program, but it requires students to think through each step in drawing a triangle, including the angles to be turned through after each length forward. Beginners often expect the angle required to be 60 degrees, because they are thinking about the internal angle in the triangle, not the external angle. The turn of 30 degrees right at the start and 30 degrees left at the end are to turn the 'turtle' from facing directly upwards at the start, and then to put it back in that position at the end of the program.

Now we have a procedure for equilateral triangles of any size, we can use it to build the triangular motif in Figure 9. With any complicated shape, it is always good practice to start by drawing a sketch on squared or isometric paper, so that relationships between side lengths and angles can be considered. This is also very helpful when the program does not do what was intended, and needs to be corrected.

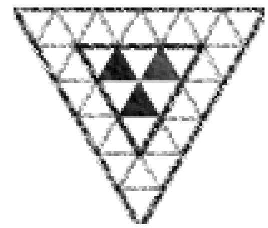

Figure 11. Triangular motif showing edge relationships

In programming the motif (Figure 11), the first thing I considered was the relationship between the side lengths of the three different sized equilateral triangles. The small triangles (shaded in Figure 11) necessarily determine the side length of the triangle directly enclosing them, but what about the outer triangle? Drawing in similar small triangles between the two enclosing triangles shows that if the side length of the small triangles is *a*, then the first enclosing triangle has a side length of 3*a*, and the outer triangle a side length of 6*a*. (Question: does this mean the side length of another enclosing triangle would be 10*a*, because the multiplying factor is a triangle number, and if so, why?)

On my first attempt at creating a program for this motif, I decided to start with the small triangles, working outwards from the centre. Later it became clear to me that it would be more useful in building up a tessellation of these motifs if the turtle started and finished at one vertex of the outer triangle. Something else that only became clear to me after a little while, was that using horizontal and vertical movements to get the cursor from one position to another was not the most efficient way to proceed on an isometric framework, and that if I needed to use multiples of 60 degree turns with the side length of the triangle, rather than calculating their heights[2].

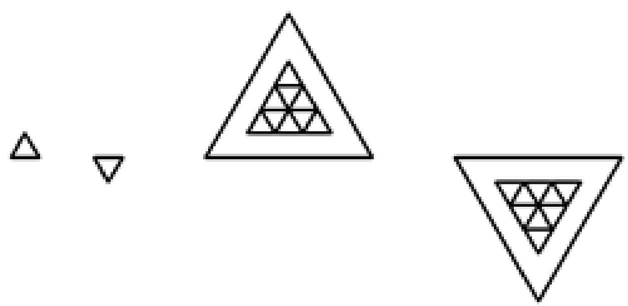

Figure 12. Executing triangleu 25, triangled 25, trimotifu 25 and trimotifd 25

My next decision was to have 'up' and 'down' versions of each of the programs, rather than having to worry about the orientation of the triangles and the motifs produced from them. Again I decided later that this was not necessary, but some of the programs given retain this aspect. Using the Procedures *to triangleu :a* and *to triangled :a*, which create equilateral triangles of side length *a*, one pointing upwards and the other pointing downwards, I created 'up' and 'down' versions of the motif, *to trimotifu :a* and *to trimotifd :a* (Figure 12).

The programs given (http://motivate.maths.org/teachers/teachers.php#topics) are not the first or even the second versions I created! As I progressed through each stage of this project, I realised that I needed to correct approaches I had taken in earlier stages. As I began to build up longer programs, producing more complex drawings, I also

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[2] Logo does recognise a sqrt () command, where the value whose square root is required is put in the brackets.

found it very helpful to tabulate my programs, giving the starting and finishing positions of the cursor, and a brief description of what the program would do.

The programs available on the webpages are the result of several hours of work, requiring considerable amounts of mathematical thinking, problem-solving and trouble-shooting. They are also work-in-progress! Working out complete sets of programs like these for the first few stages of a linear or hexagonal tessellation is time-consuming, and very frustrating for the beginner. However, working on the simpler programs is an excellent way to ensure that students understand the relationships between length and angles in geometrical motifs like this – and any misconceptions will show up in the drawings on screen! Teachers and students are welcome to use my programs at http://motivate.maths.org/teachers/teachers.php#topics. However, I should point out that these are not the only or even necessarily the most efficient ways to produce the motifs and the tessellations.

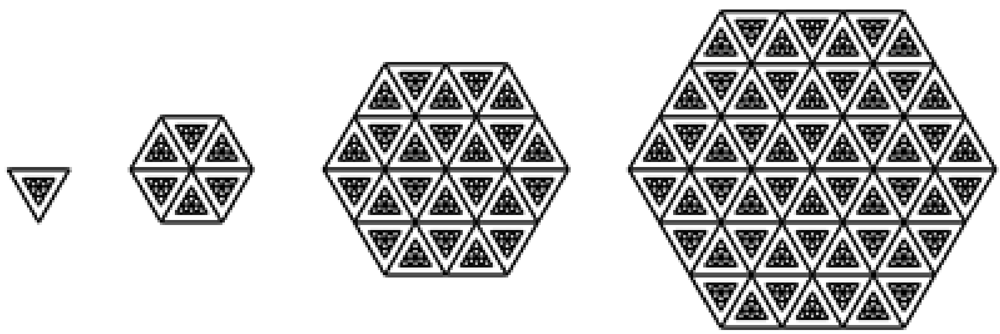

Figure 13. Hexagonal tessellation from triangular motif

A radial tessellation of the triangular motif is shown in Figure 13. Stage 1 is perhaps anomalous, and it might be considered preferable to omit this. If we consider the number sequence for the motifs which arises from this tessellation, the following table (Table 1) can be drawn up:

Table 1. Analysing the triangular tessellation

.Layer number, *n*	No. of motifs in added layer
Layer 1	1
Layer 2	6
Layer 3	18
Layer 4	30
Layer 5	?
Layer 6	?

What is happening here is that at each stage after the initial motif, we have a bigger hexagon, and the side length is increasing from 1 unit to 3 units to 5 units. It would be reasonable for students to predict that the side length will continue to increase by two units with each new layer, because an additional motif is required at each end of each edge. This helps students to predict what might happen in the next stages and to justify their predictions. This information can then be incorporated into the table (Table 2):

Table 2. Further analysis of the triangular tessellation

.Layer number, n	No. of motifs in added layer	Side length (no. of motifs)	Total no. of motifs
Layer 1	1		
Layer 2	$6 = 6 \times 1$	1	6
Layer 3	$18 = 6 \times 3$	3	24
Layer 4	$30 = 6 \times 5$	5	54
Layer 5	? $6 \times 7 = 42$	? 7	? $54 + 42 = 96$
Layer 6	? $6 \times 9 = 54$	? 9	? $96 + 54 = 150$

Students can make predictions about the number of motifs, or the side length, for successive stages of the tessellation and check them by adapting the programs to draw them. If their predictions are correct, then they can go on to generalise them into algebraic formulae.

A similar project could be undertaken with the star motif, although this has added complexity since the star cannot be used to tessellate an area without either leaving holes or including hexagons.

6. Conclusions

Apparently simple floor tilings can be used to motivate a range of mathematical discussion and investigation for a wide range of different ages and abilities. Examples of tiling patterns and lessons derived from them have been given, and the worksheets and programs used in these lessons are available online (http://motivate.maths.org/teachers/teachers.php#topics). However it is my hope that teachers reading this article will be inspired to look around them at the patterns in buildings (on walls, floors, ceilings, pavements, …) around them, and to use them in similar ways. Even the simplest of tiling patterns can be a rich source of mathematics and prompt good mathematical discussion.